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USE OF THE METHOD OF PARTICULAR SOLUTIONS IN NONLINEAR, TWO-POINT BOUNDARY-VALUE PROBLEMS PART 2 - CONTROLLED SYSTEMS

by

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Use of the Method of Particular Solutions In Nonlinear, Two-Point Boundary-Value Problems Part 2 - Controlled Systems[1]

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Abstract. The nonlinear, two-point boundary-value problem relevant to a particular controlled system is solved. First, quasilinearization techniques are used to replace the nonlinear system with one that is linear. Then, the method of particular solutions is used to solve the linear problem, and the procedure is employed iteratively. Three variations of this procedure are presented, one converging to a family of solutions and two converging to a unique solution.

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1. Introduction

In Ref. 1, Miele developed the method of particular solutions for solving the linear, two-point boundary-value problem for both an uncontrolled system and a controlled system. He treated a system of order n, subjected to p initial conditions and q final conditions, with $p + q = n$ for an uncontrolled system and $p \le n$, $q \le n$ for a controlled system. He proved that $q + 1$ particular solutions of the original, nonhomogeneous system satisfying the initial conditions but not the final conditions can be combined linearly so as to satisfy simultaneously the original, honhomogeneous system and the initial conditions, providing the sum of the constants of the linear combination is one. This relation and the q prescribed final conditions constitute a system of $q + 1$ linear algebraic equations in the $q + 1$ unknown constants.

In Ref. 2, the method of particular solutions was used to solve the two-point boundary-value problem for several nonlinear, uncontrolled systems. First, quasilinearization techniques were used to replace each nonlinear system with a system linear in the perturbations about a nominal curve (see, for example, Refs. 3 and 4). The method of particular solutions was applied to the linear system to obtain the perturbations leading to a new nominal curve; then, an iterative procedure was employed to converge to the exact solution.

In this report, the method of particular solutions is used to solve the two-point boundary value problem for a particular nonlinear, controlled system. A complete set of initial conditions and final conditions is prescribed. The object is to find one of many possible control functions which allow one to satisfy the differential system, the initial conditions, and the final conditions.

2. Control Problem

The nonlinear, controlled system considered here is described by the differential equations

$$\dot{x} - y^2 = 0, \quad \dot{y} - u = 0 \tag{1}$$

the initial conditions

$$x(0) = y(0) = 0 \tag{2}$$

and the final conditions

$$x(1) = y(1) = 1 \tag{3}$$

In these equations, the time t is the independent variable, x and y are the state variables, u is a control, and the dot denotes a derivative with respect to time. The problem is to find one particular control u(t) which gives solutions x(t) and y(t) satisfying Eqs. (1), the initial conditions (2), and the final conditions (3).

3. Solution Process

In this section, several ways for finding a satisfactory control function u(t) are presented. The first step is to linearize Eqs. (1) about a prescribed nominal curve $x_*(t)$, $y_*(t)$, $u_*(t)$ to obtain the perturbation equations

$$\begin{aligned} &\delta\dot{x} - 2y_* \delta y + (\dot{x} - y^2)_* = 0 \\ &\delta\dot{y} - \delta u + (\dot{y} - u)_* = 0 \end{aligned} \tag{4}$$

Here, the symbols δx, δy, δu, denote the perturbations of x, y, u at a constant station t, that is,

$$\begin{aligned} \delta x &= x(t) - x_*(t) \\ \delta y &= y(t) - y_*(t) \\ \delta u &= u(t) - u_*(t) \end{aligned} \tag{5}$$

Since the nominal curve is always chosen to satisfy Eqs. (2)-(3), the initial conditions for the system (4) are

$$\delta x(0) = \delta y(0) = 0 \tag{6}$$

and the final conditions are

$$\delta x(1) = \delta y(1) = 0 \tag{7}$$

Having linearized Eqs. (1), we apply the method of particular solutions. Since $q + 1 = 3$, three particular solutions are required and are designated with the subscripts 1, 2, 3, respectively. These solutions are first obtained by integrating Eqs. (4) subject to (6) and three different control perturbations $\delta u(t)$ and then combined so as to satisfy the final conditions (7). Several ways to choose the control perturbations are shown below.

Method A. The nominal curve is chosen to be

$$x_* = t, \quad y_* = t, \quad u_* = t \tag{8}$$

This curve does not satisfy the differential equations (1), but it satisfies the initial conditions (2) and the final conditions (3). As noted before, three particular solutions of the linear system (4) are needed for the perturbation functions. In the first integration, we employ the initial conditions

$$\delta x_1(0) = \delta y_1(0) = 0 \tag{9}$$

and the control perturbation

$$\delta u_1 = t(1 - t) \tag{10}$$

to obtain the state variable perturbations

$$\delta x_1(t), \qquad \delta y_1(t) \tag{11}$$

In the second integration, we employ the initial conditions

$$\delta x_2(0) = \delta y_2(0) = 0 \tag{12}$$

and the control perturbation

$$\delta u_2 = t^2(1 - t) \tag{13}$$

to obtain the state variable perturbations

$$\delta x_2(t), \qquad \delta y_2(t) \tag{14}$$

In the third integration, we employ the initial conditions

$$\delta x_3(0) = \delta y_3(0) = 0 \tag{15}$$

and the control perturbation

$$\delta u_3 = t^3(1 - t) \tag{16}$$

to obtain the state variable perturbations

$$\delta x_3(t), \qquad \delta y_3(t) \tag{17}$$

Next, we form the linear combinations

$$\begin{aligned} \delta x &= k_1 \delta x_1 + k_2 \delta x_2 + k_3 \delta x_3 \\ \delta y &= k_1 \delta y_1 + k_2 \delta y_2 + k_3 \delta y_3 \\ \delta u &= k_1 \delta u_1 + k_2 \delta u_2 + k_3 \delta u_3 \end{aligned} \tag{18}$$

and inquire whether they can satisfy the differential equations (4), the initial conditions (6), and the final conditions (7). As shown in Ref. 1, the functions (18) satisfy (4) and (6) provided

$$k_1 + k_2 + k_3 = 1 \tag{19}$$

and (7) provided

$$\begin{aligned} k_1 \delta x_1(1) + k_2 \delta x_2(1) + k_3 \delta x_3(1) &= 0 \\ k_1 \delta y_1(1) + k_2 \delta y_2(1) + k_3 \delta y_3(1) &= 0 \end{aligned} \tag{20}$$

Equations (19)-(20) are a system of three algebraic equations which determine the constants k_1, k_2, k_3. Once the perturbation functions are known, the approximate trajectory of the system is given by

$$\begin{aligned} x &= x_* + \delta x \\ y &= y_* + \delta y \\ u &= u_* + \delta u \end{aligned} \tag{21}$$

and, in this way, the first iteration is completed. Next, the functions (21) are employed as the nominal functions for the second iteration, and the process is repeated.

Computations were performed with an IBM 7040 computer, and the results are shown in Figs. 1-3, in which n is the iteration number. The curve $n = 0$ is the nominal curve (8), the curve $n = 1$ is that obtained after one iteration, and so on. The curve $n = 4$ is deleted because it lies so close to the curve $n = 3$ that the relative differences cannot be seen in the scale of Figs. 1-3. Convergence to a solution is rapid in that the modulus of the error functions

$$E_1 = \dot{x} - y^2, \qquad E_2 = \dot{y} - u \tag{22}$$

is less than 10^{-4} at every station t after five iterations.

In the sixth iteration and subsequent iterations, the functions x, y, u continue to change at a small but approximately constant rate, while the magnitude of E_1 and E_2 remains about the same. In other words, Method A does not generate a unique solution but, for $n \geq 5$, a family of solutions. In this connection, the curve $n = 10$ is shown in Figs. 1-3. The nonuniqueness is obvious if one examines Eqs. (5-3), (10), (13), (16), (18-3), and (19). Clearly, a unique solution is possible if, and only if, $\delta u(t) \to 0$ as $n \to \infty$. However, the functions $\delta u_1(t)$, $\delta u_2(t)$, $\delta u_3(t)$ are identical for each iteration and their weighting coefficients k_1, k_2, k_3 cannot all be small because of (19).

Method B. This method is identical with Method A, except that the control perturbations are defined as

$$\delta u_1 = t(1 - t)/n^2$$

$$\delta u_2 = t^2(1 - t)/n^2 \qquad (23)$$

$$\delta u_3 = t^3(1 - t)/n^2$$

where n denotes the generic iteration. Since Eqs. (23) yield smaller control perturbations as successive iterations are performed, one must expect a unique solution.

Computations were performed with an IBM 7040 computer. Convergence to a solution is rapid in that the modulus of the error functions (22) is less than 10^{-6} at every station t after five iterations. The sixth iteration and subsequent iterations produce increasingly smaller changes in the functions x, y, u, indicating convergence to a unique solution (see Figs. 4-6). This unique solution is almost identical with the curve n = 3 of Figs. 1-3.

Method C. This method is identical with Method A, except that the control perturbations are defined as

$$\delta u_1 = \int_0^t E_1 dt, \quad \delta u_2 = \int_0^t E_1^2 dt, \quad \delta u_3 = \int_0^t E_2 dt \qquad (24)$$

Since Eqs. (24) yield smaller control perturbations as successive iterations are performed, one must expect a unique solution.

Computations were performed with an IBM 7040 computer. Convergence to a solution is rapid in that the modulus of the error functions (22) is less than 10^{-5} at every station t after five iterations. The sixth iteration and subsequent iterations produce increasingly smaller changes in the functions x, y, u, indicating convergence to a unique solution (see Figs. 7-9).

4. Conclusions

Two mathematical techniques, quasilinearization and the method of particular solutions, are combined to solve the nonlinear, two-point boundary-value problem relevant to a particular controlled system. Specifically, the nonlinearity of the problem is removed by quasilinearization, and the resulting linear, two-point boundary-value problem is solved by the method of particular solutions.

Three different methods for finding a suitable control function are used. In all of these, a nominal curve not satisfying the differential equations but satisfying the boundary conditions is chosen. Then, a control perturbation is applied, and the state variable perturbations are found.

In Method A, control perturbations independent of the iteration number are employed. In Method B, control perturbations inversely proportional to the square of the iteration number are used. In Method C, control perturbations proportional to the powers of the integral of the errors committed in the differential equations are employed. While Method A does not lead to a unique solution, Methods B and C do lead to a unique solution. Method B appears to be the best for the particular problem under consideration.

In conclusion, the combination of quasilinearization with the method of particular solutions can be a powerful tool in solving nonlinear, two-point boundary-value problems relevant to controlled systems. Provided the initial guess used in the iteration procedure is chosen with discretion, convergence to a solution is quite rapid, and the accuracy of the solution is limited only by the integration step size and the integration technique employed.

References

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List of Captions

Fig. 1 The function x(t).

Fig. 2 The function y(t).

Fig. 3 The function u(t).

Fig. 4 The function x(t).

Fig. 5 The function y(t).

Fig. 6 The function u(t).

Fig. 7 The function x(t).

Fig. 8 The function y(t).

Fig. 9 The function u(t).

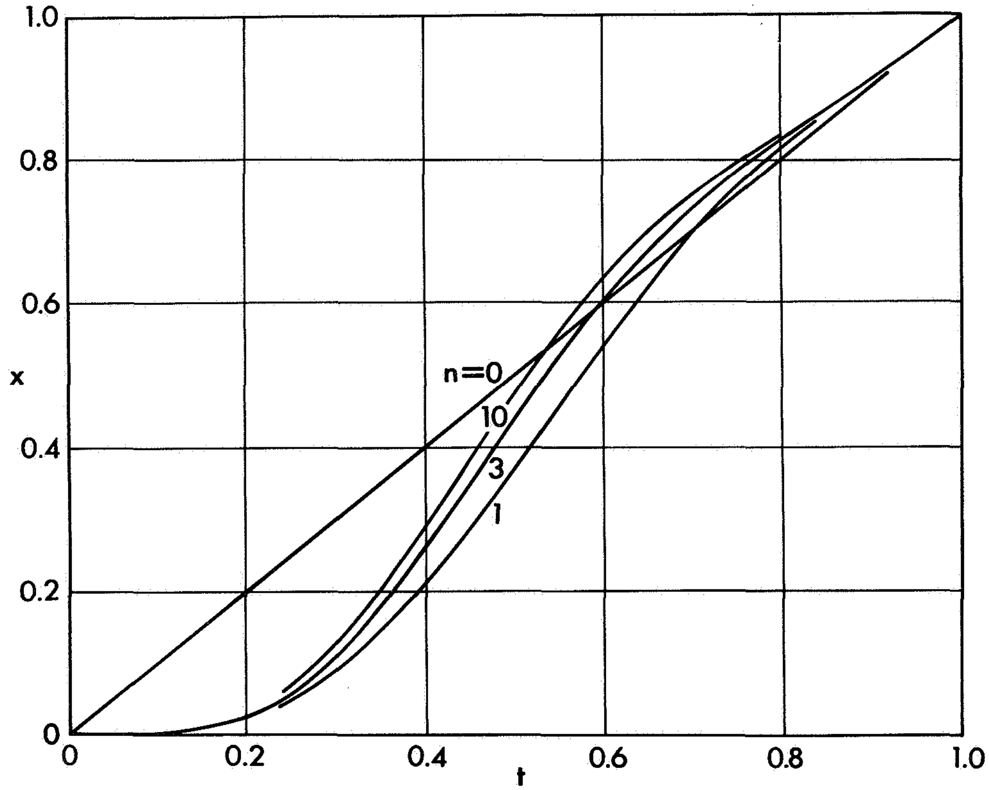

Fig. 1   The function x(t).

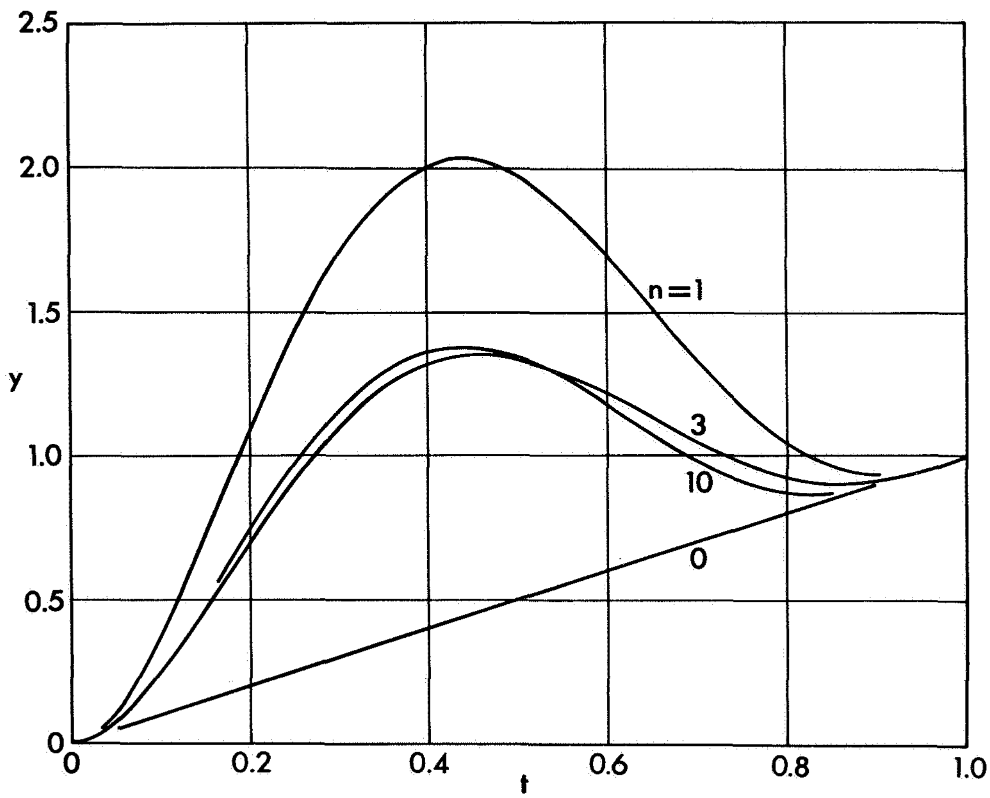

Fig. 2   The function y(t).

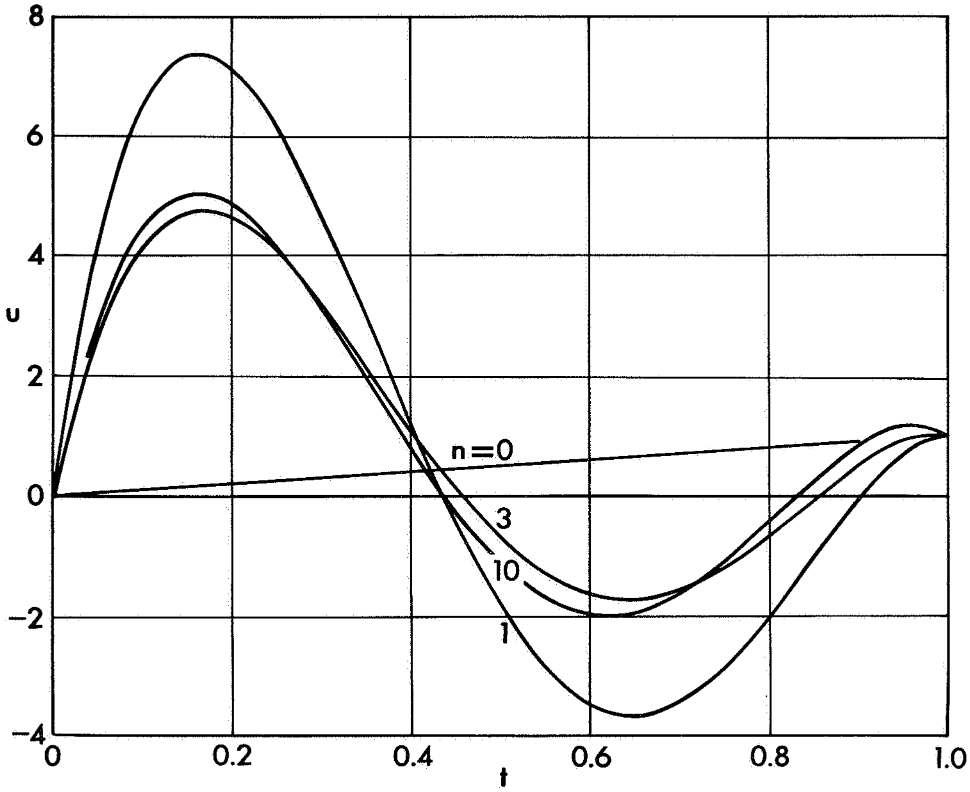

Fig. 3   The function u(t).

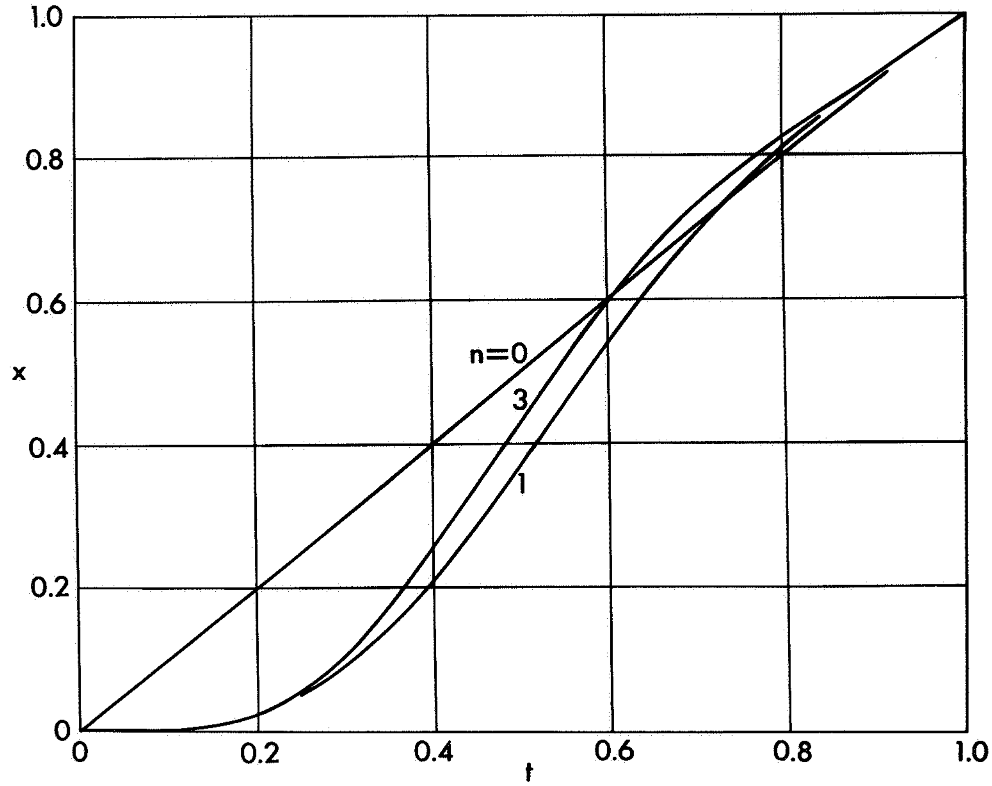

Fig. 4 The function x(t).

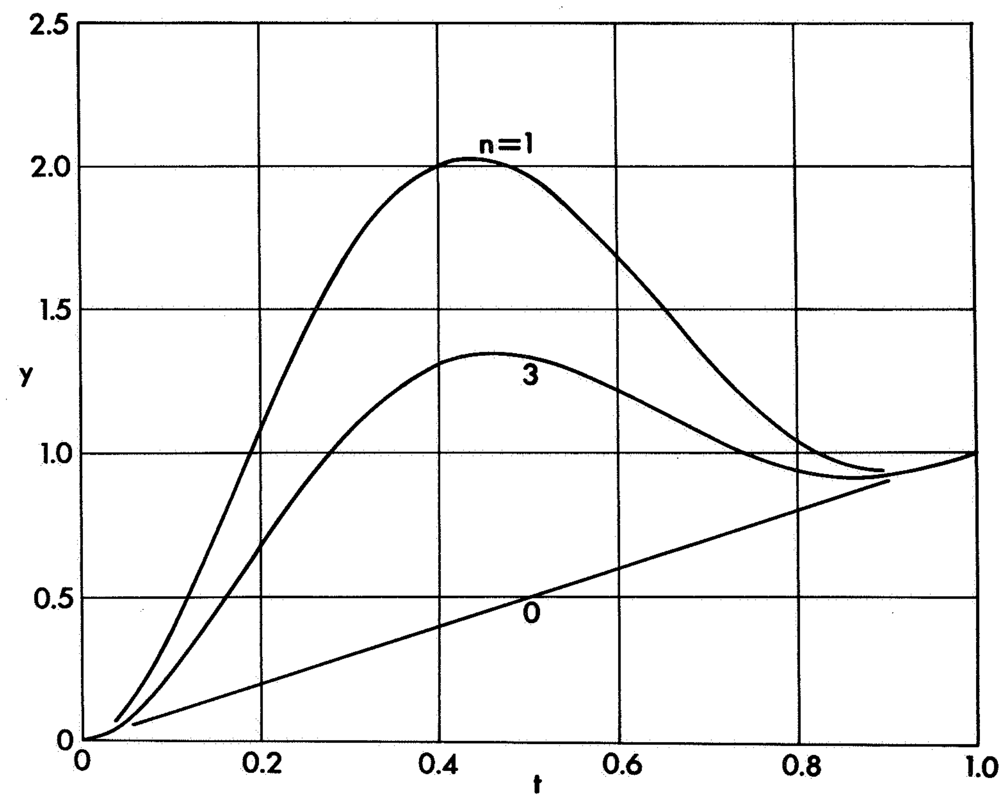

Fig. 5 The function y(t).

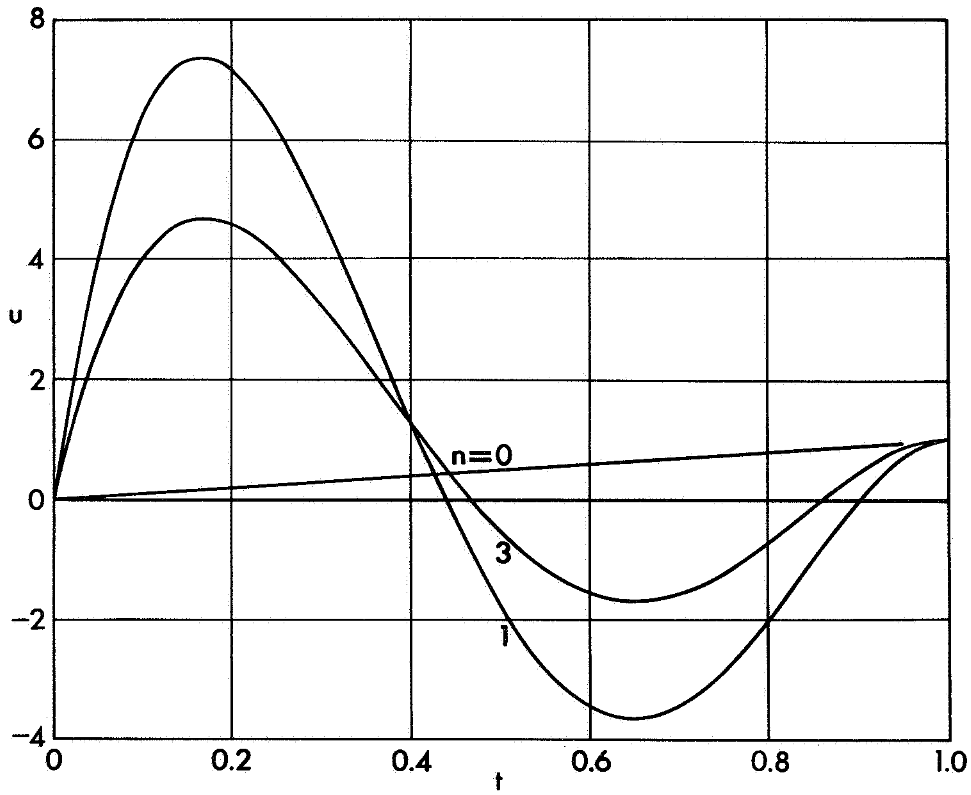

Fig. 6 The function u(t).

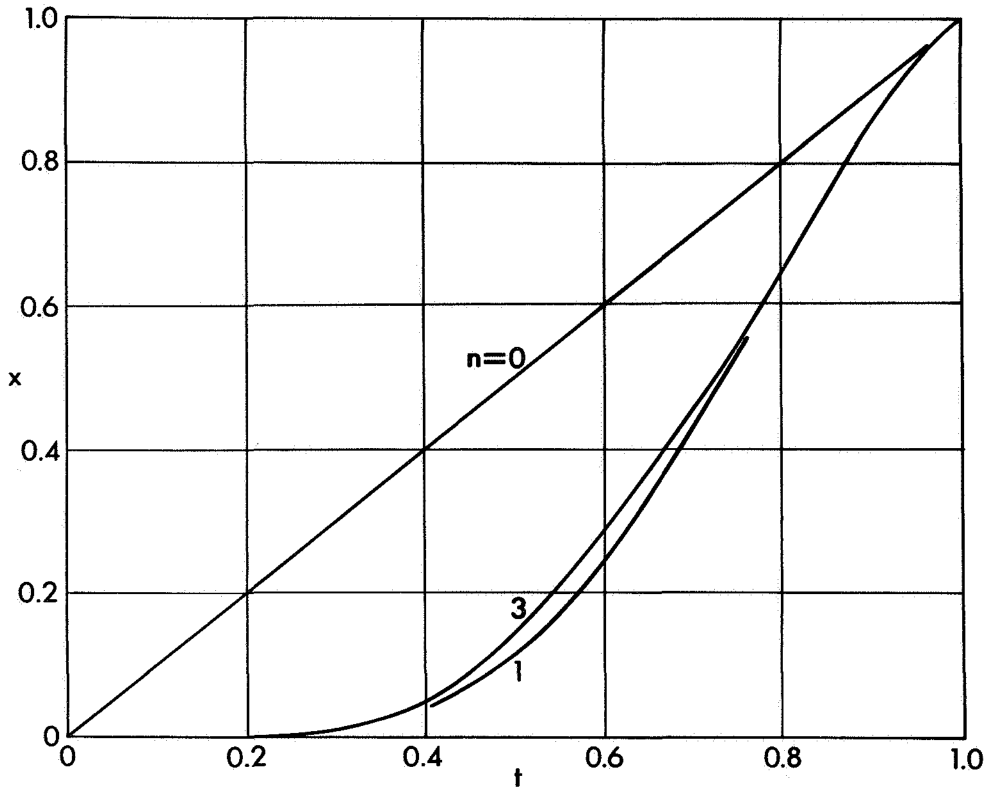

Fig. 7 The function x(t).

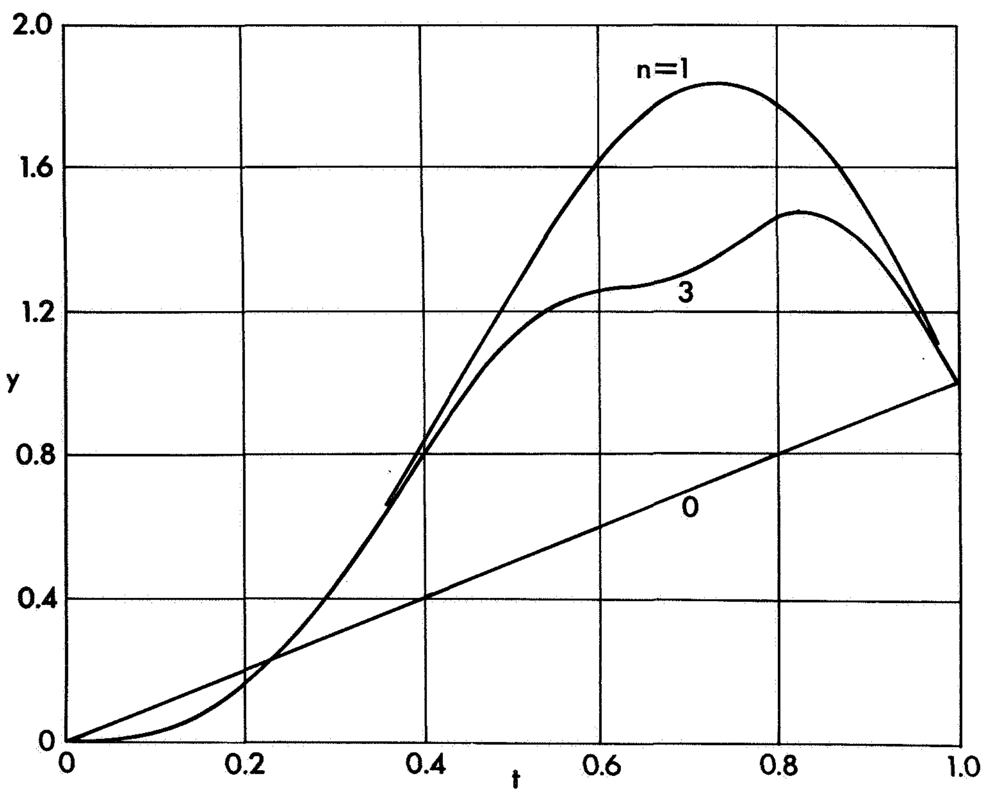

Fig. 8 The function y(t).

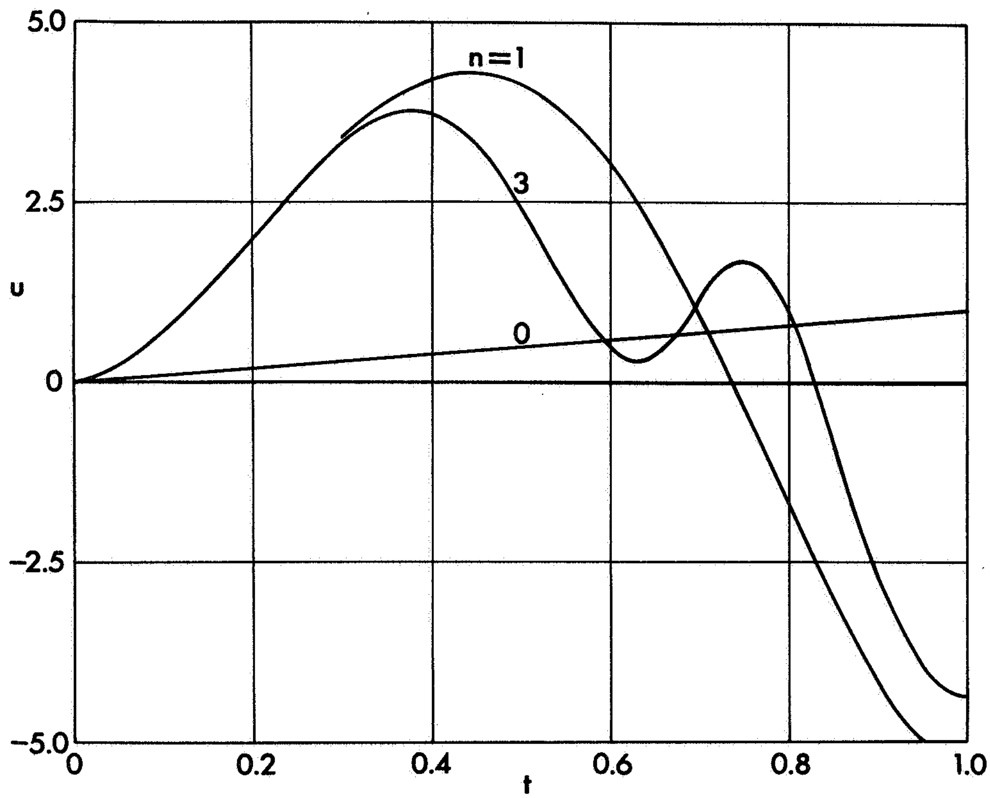

Fig. 9 The function u(t).